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A c -theorem for entanglement entropy

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Abstract

The combination of the Lorentz symmetry and the strong subadditive property of the entropy leads to a c -theorem for the entanglement entropy in (1+1) dimensions. We present a simple derivation of this theorem and compare the associated c -functions with the Zamolodchikov's ones for the case of free fields. We discuss the various difficulties which are obstacles to the naive generalizations of the entropic c -theorem to higher dimensions.

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1. Entanglement entropy

Given a subset A of the space the Hilbert space of states can be decomposed as a tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{-A}$ of the states localized in A and in the complementary region $-A$. If we take as a global state the vacuum or the fundamental state $|\Psi\rangle$ of the system, the state ρ_A relevant to the physics in A corresponds to the partial trace over the Hilbert space \mathcal{H}_{-A} . This gives the local reduced density matrix

$$\rho_A = \text{tr}_{\mathcal{H}_{-A}} |\Psi\rangle\langle\Psi|. \quad (1)$$

Its von Neumann entropy is usually called entanglement or geometric entropy,

$$S(A) = -\text{tr}(\rho_A \log \rho_A). \quad (2)$$

It is generally different from zero since the global vacuum state contains correlations in the bipartite system. This implies that the information cannot be completely recovered with the only knowledge of the state in A and leads to a mixed state for the local region.

The entanglement entropy was proposed originally to give a statistical explanation to the intriguing entropy of the black holes [1]. In this sense it is still one of the most physically appealing candidates. However the presence of ultraviolet divergences suggests that the role of quantum gravity should be relevant in producing a finite entropy. In a series of recent papers a definite conjecture for the meaning of the entanglement entropy in the context of the AdS-CFT duality has been given [2].

More recently, partially motivated by advances in quantum information theory and the density matrix renormalization group method, different measures of information and entanglement have attracted the attention of several authors working in condensed matter and low dimensional systems. In this context it was discovered that the description of several phenomena can be translated to the language of the entanglement properties of fundamental states. In particular, this includes quantum phase transitions [3], topological order [4], and, as will be reviewed in this paper, the renormalization group irreversibility in two dimensions [5, 6].

2. Entanglement entropy and quantum field theory

From the point of view of quantum field theory (QFT) the entanglement entropy $S(A)$ can be considered as a non-local variable which is definable for any theory disregarding the field content. In this sense it can be identified as a natural tool to investigate general properties of QFT. In fact, the entanglement entropy satisfies very remarkable non-perturbative relations (see for example [7])

$$S(A) = S(-A), \quad (3)$$

$$S(A) + S(B) \geq S(A \cup B) + S(A \cap B). \quad (4)$$

The first one follows from the purity of the vacuum state, while the second, called strong subadditivity (SSA), is a general and very important property of the entropy for multipartite systems.

From a technical point of view, the entanglement entropy $S(A)$ is proportional to the variation of the Euclidean free energy with respect to the introduction of a small conical singularity at the boundary of A [8, 9]. Explicitly we have

$$S(A) = \lim_{\alpha \rightarrow 1} \left(1 - \frac{d}{d\alpha} \right) \log Z(A, \alpha), \quad (5)$$

where $Z(A, \alpha)$ is the Euclidean partition function of the theory on a space with conical singularity of angle $2\pi\alpha$ at the boundary of A .

The entanglement entropy in the continuum theory is ultraviolet divergent. However, it has a nice geometrical structure of divergences. In d spatial dimensions we have [10]

$$S(V) = g_{d-1}[\partial V]\epsilon^{-(d-1)} + \dots + g_1[\partial V]\epsilon^{-1} + g_0[\partial V] \log(\epsilon) + S_0(V), \quad (6)$$

where $S_0(V)$ is a finite part, ϵ is a short distance cutoff, and the g_i are local and extensive functions on the boundary ∂V , which are homogeneous of degree i . The leading divergent term coefficient $g_{d-1}[\partial V]$ is then proportional to the $(d-1)$ power of the size of V , and this is usually referred to as the area law for the entanglement entropy [11]. However, g_i for $i > 0$ depends on the regularization procedure and g_{d-1} is not proportional to the area if this later is not rotational invariant. These terms are not physical within QFT since they are not related to continuum quantities.

Universal quantities are present in $S(A)$ however. In particular the mutual information

$$I(A, B) = S(A) + S(B) - S(A \cup B) \quad (7)$$

between two non-intersecting regions is universal since the boundary terms get subtracted away [5]. In fact, a general theorem based on the strong subadditive property and the Lorentz covariance [7] establishes that if the entanglement entropy is finite, the mutual information is identically zero for a relativistic QFT. The presence of a non-zero mutual information implies that the entropy cannot be made finite through contributions coming from the physics at the high energy sectors (within QFT).

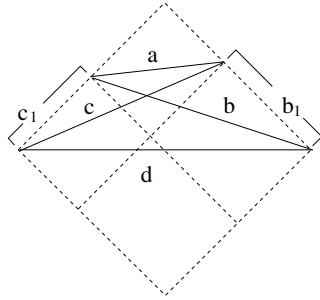


Figure 1. Time is the vertical axis, the horizontal axis is the spatial coordinate x and the null lines are drawn at 45° . The causal domain of dependence (diamond shaped set drawn with dashed lines) corresponding to the spatial intervals b, c have intersection given by the domain of dependence of a and union (followed by causal completion) given by the domain of dependence of d .

3. The c -theorem

The statement of the c -theorem in $(1 + 1)$ dimensions [12] can be expressed in two equivalent ways.

- (a) There is a universal dimensionless function of the couplings $c(\{\lambda_i\})$ in the theories space in $(1 + 1)$ dimensions which is non-increasing along the renormalization group trajectories and stationary at the fixed points, where it takes a finite value proportional to the Virasoro central charge C_V .
- (b) For any theory in $(1+1)$ dimensions there is a universal dimensionless function of a distance $c(r)$ which is non-increasing under dilatations and takes a finite value proportional to C_V at the fixed points.

An interesting interpretation which follows from the formulation (a) is that the c -function measures some kind of entropy related to the information which is lost during the renormalization group transformations. However, if we take this too literally, we could conclude that the theorem should also be valid outside the domain of unitary and relativistic theories, where it actually does not hold. In contrast, the formulation (b) does not refer to the renormalization group and show the c -theorem as a property of the continuum theory, disregarding the procedure used to define it.

The Zamolodchikov’s proof of the c -theorem is given in the Euclidean formulation of the two-dimensional QFT, and involves the covariance of the energy momentum tensor correlators plus the reflexion positivity property. Here we show a very simple alternative proof based on the strong subadditive property of the entanglement entropy and the Lorentz invariance in a real spacetime formulation [5].

Consider two intervals of lengths b and c relatively boosted to each other and located as shown in figure 1. The causal domain of dependence of these intervals (causal shadow) have intersection and causal union given by those corresponding to the two intervals of length a and d . These sets appear in the small side of the relativistic version of the strong subadditive inequality between b and c [7]. Then, we have¹

$$S(c) + S(b) \geq S(a) + S(d). \tag{8}$$

¹ To be explicit, this relation is just the standard SSA inequality on the (non-timelike) surface $c_1 \cup a \cup b_1$ (see figure 1), which gives $S(c_1 \cup a) + S(a \cup b_1) \geq S(a) + S(c_1 \cup a \cup b_1)$, followed by the identification of $S(c_1 \cup a) = S(c)$, $S(a \cup b_1) = S(b)$, and $S(c_1 \cup a \cup b_1) = S(d)$, which is a consequence of the unitarity of the causal evolution.

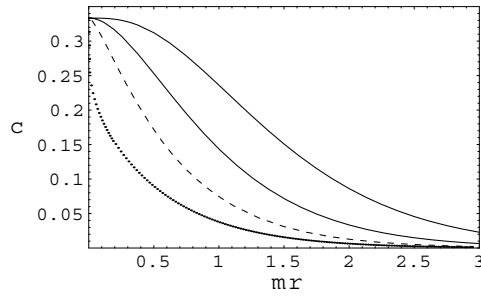


Figure 2. From top to bottom: one third of the Zamolodchikov c -functions for a real scalar and a Dirac field, and entropic c -functions for a Dirac (dashed curve) and a real scalar field (dotted curve).

The relativistic geometry gives the simple relation

$$ad = cb, \tag{9}$$

which can be rewritten as

$$c = \lambda a, \quad d = \lambda b, \quad \lambda = \frac{c}{a} = \frac{d}{b} \geq 1. \tag{10}$$

Therefore (8) gives

$$S(b) - S(a) \geq S(\lambda b) - S(\lambda a). \tag{11}$$

This means that the difference of the entropies $S(b) - S(a)$, with $b \geq a$ is non-increasing under dilatations. It is also dimensionless and universal since according to (6) in two dimensions the divergent term for an interval is a constant (independent of the interval size) proportional to $\log(\epsilon)$. At the conformal fixed points the entropy can be explicitly evaluated [9], and is given by

$$S(r) = \frac{C_V}{3} \log(r/\epsilon). \tag{12}$$

Thus, at the fixed points $S(b) - S(a) = (C_V/3) \log(b/a) \geq 0$. We conclude that $S(b) - S(a)$ is a c -function for any $b > a$.

An entropic c -function depending on a single parameter (and containing all the information in $S(b) - S(a)$) can be defined by

$$c(r) = r \frac{dS(r)}{dr}, \tag{13}$$

which is dimensionless, universal, positive, and according to (11) satisfies

$$c'(r) = rS''(r) + S'(r) \leq 0. \tag{14}$$

We have computed the entropic c -functions for free fermions and bosons numerically in a lattice and found analytically short and long distance expansions using (5) [13]. The analytic results involve sine-Gordon model correlators of exponential operators which are given in terms of Painlevé V differential equations. As shown in figure 2 these are quite different from the Zamolodchikov's ones. The leading short and long distance terms read

$$c_D(t) \sim \frac{1}{3} - \frac{1}{3}t^2 \log^2(t) \quad \text{for } t \ll 1; \quad c_D(t) \sim \frac{1}{2}tK_1(2t) \quad \text{for } t \gg 1, \tag{15}$$

$$c_S(t) \sim \frac{1}{3} + \frac{1}{2\log(t)} \quad \text{for } t \ll 1; \quad c_S(t) \sim \frac{1}{4}tK_1(2t) \quad \text{for } t \gg 1, \tag{16}$$

where $t = mr$, m is the field mass, and $c_D(t)$ and $c_S(t)$ are the entropic c -functions corresponding to a free Dirac and a real scalar field respectively.

4. Is there an entropic c -theorem in more dimensions?

Though several authors have made much progress in the direction of extending the c -theorem to higher dimensions using correlators of the energy momentum tensor, no definitive result has been obtained so far [14]. Thus, it is tempting to try the present method in more dimensions, exploiting the Lorentz symmetry and the strong subadditive inequality on the geometric entropy. However, one has to bear in mind that no compelling physical principle guarantees that a version of the theorem has to hold in $d \geq 2$ [15]. We have essayed different possibilities without success, since several obstacles show up in higher dimensions. What follows is a list of some of them. They may be avoided in de Sitter space but at the expense of introducing a dimensionfull parameter (the spacetime radius) which impedes the proof of the theorem.

- (a) *The shape problem*: a generalization of the construction on figure 1 requires to select a specific shape for the set replacing the interval in two dimensions. The first obstacle is then that in $d \geq 2$ the intersections and unions of sets with a given shape are in general of a different shape. Also, the SSA relation between them in general contains divergent terms on each side which do not cancel due to the new surface features contained in the intersection and the union. This puts difficulties in the gain of useful information from the SSA relation.
- (b) *The commutativity problem*: there are certain shapes for which the previous problem does not appear, for example two rectangles formed by moving one of them along one of its sides. However, if the rectangles are boosted to each other (this is essential in (11)) there are pieces of the boundary of one of the rectangles which are at a timelike distance from the other: technically they do not commute in the algebra of causal sets, and the SSA relation cannot be used [7]. We can avoid this problem by using the limit of rectangles with one large side. However, in this case we have to consider intensive quantities with respect to the large side, the corresponding c -functions are dimensionfull, and cannot be used to prove the theorem.
- (c) *The order of the divergences*: this is perhaps the deepest problem encountered. Because of its nature, the SSA relation when used between infinitesimally displaced sets, allow us to obtain inequalities involving only second-order derivatives of the entropy. However, in d spatial dimensions we have general divergent terms with $(d - 1)$ dimensions. Thus, the local SSA relation cannot give inequalities for universal quantities if $d \geq 2$.
- (d) *Logarithmic terms*: the mutual information $I(A, B)$ is universal, dimensionless and increasing with the size of A and B . Thus, $I(A, B)$ is decreasing under dilatations for star-shaped non-intersecting sets A and B . However $I(A, B)$ diverges logarithmically at critical points due to the logarithmic terms induced by the vertices [10].

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